A Level Further Maths transition pack

You should complete the A level Maths transition pack. In addition to that taking Further Maths will require additional problem solving skills and an appreciation of number and algebra. The UK Maths Trust Senior Maths challenge is a rich source of interesting maths problems aimed at students in year 13 and below.

Have a go at these two past maths challenge papers – your are not expected to be able to answer every question!

More papers can be found at the UKMT website https://ukmt.org.uk/senior-challenges/senior-mathematical-challenge

UK Maths Trust

Senior Mathematical Challenge

Tuesday 3 October 2023

Organised by the United Kingdom Mathematics Trust

supported by Overleaf

Candidates must be full-time students at secondary school or FE college.
England & Wales: Year 13 or below
Scotland: S6 or below
Northern Ireland: Year 14 or below

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **90 minutes**. No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank paper for rough working is allowed; **squared paper, calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil**. Mark at most one of the options, A, B, C, D, or E, on the Answer Sheet for each question. Do not mark more than one option.
5. **Do not expect to finish the whole paper in the time allowed**. The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
6. **Scoring rules**: All candidates start with 25 marks; 0 marks are awarded for each question left unanswered; 4 marks are awarded for each correct answer; 1 mark is deducted for each incorrect answer (to discourage guessing).
7. **Your Answer Sheet will be read by a machine**. Do not write or doodle on the sheet except to mark your chosen options. The machine will read all black pencil markings even if they are in the wrong places. If you mark the sheet in the wrong place, doodle, or leave bits of eraser stuck to the page, the machine will interpret the mark in its own way, or reject the answer sheet.
8. **The questions on this paper are designed to challenge you to think, not to guess**. You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.
9. To accommodate candidates sitting at other times, please do not discuss the paper on the internet until **08:00 BST on Thursday 5 October**.

Enquiries about the Senior Mathematical Challenge should be sent to:

challenges@ukmt.org.uk

www.ukmt.org.uk

1. What is the value of $\sqrt{\dfrac{2023}{2+0+2+3}}$?

A 13 B 15 C 17 D 19 E 21

2. What is the difference between one-third and 0.333?

A 0 B $\dfrac{3}{1000}$ C $\dfrac{1}{3000}$ D $\dfrac{3}{10000}$ E $\dfrac{1}{30000}$

3. The base of a triangle is increased by 20% and its height is decreased by 15%.

What happens to its area?

A It decreases by 3% B It remains the same C It decreases by 2%
D It increases by 2% E It increases by 5%

4. In 2016, the world record for completing a 5000m three-legged race was 19 minutes and 6 seconds. It was set by Damian Thacker and Luke Symonds in Sheffield.

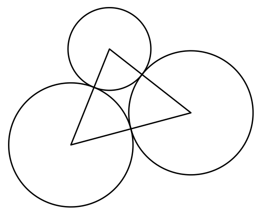
What was their approximate average speed in km/h?

A 10 B 12 C 15 D 18 E 25

5. Three circles with radii 2, 3 and 3 touch each other, as shown in the diagram.

What is the area of the triangle formed by joining the centres of these circles?

A 10 B 12 C 14 D 16 E 18



6. How many lines of three adjacent cells can be chosen from this grid, horizontally, vertically or diagonally, such that the sum of the numbers in the three cells is a multiple of three?

A 30 B 24 C 18 D 12 E 6

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

7. A sequence begins 2023, 2022, 1, ... After the first two terms, each term is the positive difference between the previous two terms.

What is the value of the 25^{th} term?

A 2010 B 2009 C 2008 D 2007 E 2006

8. What is the value of $99(0.\dot{4}\dot{9} - 0.\dot{4})$?

A 5 B 4 C 3 D 2 E 1

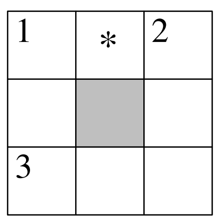
9. When completed correctly, the cross number is filled with four three-digit numbers.

What digit is *?

A 0 B 1 C 2
D 4 E 6

Across
1. A square
3. A fourth power

Down
1. Twice a fifth power
2. A cube



10. How many of the numbers 6, 7, 8, 9, 10 are factors of the sum $2^{2024} + 2^{2023} + 2^{2022}$?

A 1 B 2 C 3 D 4 E 5

11. Wenlu, Xander, Yasser and Zoe make the following statements:

Wenlu: "Xander is lying." Xander: " Yasser is lying."

Yasser: "Zoe is telling the truth." Zoe: "Wenlu is telling the truth."

What are the possible numbers of people telling the truth?

A 1 or 2 B 1 or 3 C 2 D 2 or 3 E 3

12. The greatest power of 7 which is a factor of 50! is 7^k $(n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n)$.

What is k?

A 4 B 5 C 6 D 7 E 8

13. $PQRST$ is a regular pentagon. The point U lies on ST such that $\angle QPU$ is a right angle.

What is the ratio of the interior angles in triangle PUT?

A 1 : 3 : 6 B 1 : 2 : 4 C 2 : 3 : 4 D 1 : 4 : 8 E 1 : 3 : 5

14. The points P $(d, -d)$ and Q $(12 - d, 2d - 6)$ both lie on the circumference of the same circle whose centre is the origin.

What is the sum of the two possible values of d?

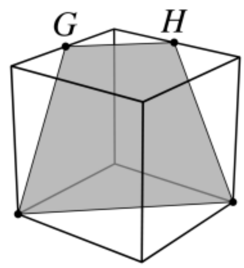
A −16 B −4 C 4 D 8 E 16

15. In Bethany's class of 30 students, twice as many people played basketball as played football. Twice as many played football as played neither.

Which of the following options could have been the number of people who played both?

A 19 B 14 C 9 D 5 E 0

16. G and H are midpoints of two adjacent edges of a cube. A trapezium-shaped cross-section is formed cutting through G, H and two further vertices, as shown. The edge-length of the cube is $2\sqrt{2}$.



What is the area of the trapezium?

A 9 B 8 C $4\sqrt{5}$ D $4\sqrt{3}$ E $4\sqrt{2}$

17. The number $M = 124563987$ is the smallest number which uses all the non-zero digits once each and which has the property that none of the pairs of its consecutive digits makes a prime number. For example, the 5th and 6th digits of M make the number 63 which is not prime. N is the largest number which uses all the non-zero digits once each and which has the property that none of the pairs of its consecutive digits makes a prime number.

What are the 5th and 6th digits of N?

A 6 and 3 B 5 and 4 C 5 and 2 D 4 and 8 E 3 and 5

18. How many solutions are there of the equation $1 + 2\sin X - 4\sin^2 X - 8\sin^3 X = 0$ with $0° < X < 360°$?

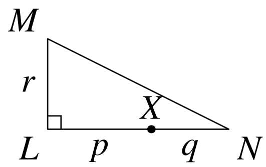
A 1 B 2 C 4 D 6 E 8

19. The expression $\dfrac{7n+12}{2n+3}$ takes integer values for certain integer values of n.

What is the sum of all such integer values of the expression?

A 4 B 8 C 10 D 12 E 14

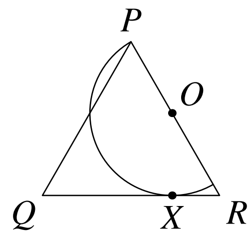
20. Triangle LMN represents a right-angled field with $LM = r$, $LX = p$ and $XN = q$. Jenny and Vicky walk at the same speed in opposite directions along the edge of the field, starting at X at the same time. Their first meeting is at M.



Which of these expressions gives q in terms of p and r?

A $\dfrac{p}{2} + r$ B $\sqrt{p^2 + r^2} + \dfrac{p}{2}$ C $\dfrac{pr}{2p + r}$ D $\dfrac{p}{2}$ E 1

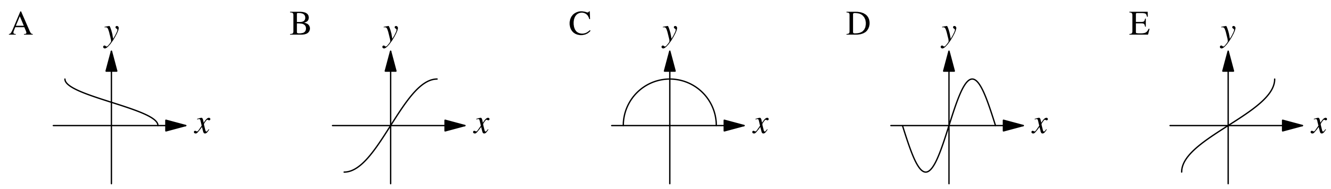
21. Triangle PQR is equilateral. A semicircle with centre O is drawn with its diameter on PR so that one end is at P and the curved edge touches QR at X. The radius of the semicircle is $\sqrt{3}$.



What is the length of QX?

A $\sqrt{3}$ B $2 - \sqrt{3}$ C $2\sqrt{3} - 1$ D $1 + \sqrt{3}$ E $2\sqrt{3}$

22. Which diagram could be a sketch of the curve $y = \sin(\cos^{-1} x)$?

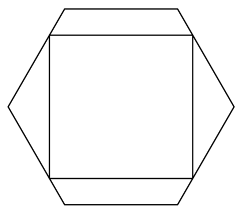


23. The length of a rectangular piece of paper is three times its width. The paper is folded so that one vertex lies on top of the opposite vertex, thus forming a pentagonal shape.

What is the area of the pentagon as a fraction of the area of the original rectangle?

A $\dfrac{2}{3}$ B $\dfrac{11}{16}$ C $\dfrac{12}{17}$ D $\dfrac{13}{18}$ E $\dfrac{14}{19}$

24. A square has its vertices on the edges of a regular hexagon. Two of the edges of the square are parallel to two edges of the hexagon, as shown in the diagram. The sides of the hexagon have length 1.



What is the length of the sides of the square?

A $\dfrac{5}{4}$ B $3 - \sqrt{3}$ C $\dfrac{4}{3}$ D $\sqrt{2}$ E $\dfrac{3}{2}$

25. What is the area of the part of the xy-plane within which $x^3y^2 - x^2y^2 - xy^4 + xy^3 \geq 0$ and $0 \leq x \leq y$?

A $\dfrac{1}{4}$ B $\dfrac{1}{2}$ C 1 D 2 E 4

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Tuesday 3 October 2023

For reasons of space, these solutions are necessarily brief.

There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation.

There is also a version of this document available on the UKMT website which includes each of the questions alongside its solution:

www.ukmt.org.uk

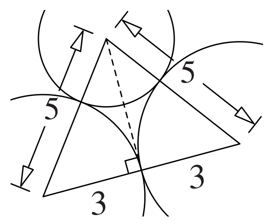
1. **C** The prime factorisation of 2023 is $7 \times 17 \times 17$ so $\sqrt{\dfrac{2023}{2+0+2+3}} = \sqrt{\dfrac{2023}{7}} = \sqrt{17^2} = 17$.

2. **C** The difference between one third and 0.333 is $\dfrac{1}{3} - \dfrac{333}{1000} = \dfrac{1000-999}{3000} = \dfrac{1}{3000}$.

3. **D** The new area = the old area $\times 1.2 \times 0.85$ = the old area $\times 1.02$. This represents a 2% increase.

4. **C** The world record of 5000 m in 19 minutes and 6 seconds $\simeq$ 5000 m in 20 minutes = 15000 m in 60 minutes = 15000 m in an hour = 15 km/h.

5. **B** The triangle formed by joining the centres of the circles is isosceles, so splitting it along its line of symmetry gives us two right-angled triangles each with a base of 3 and a hypotenuse of 5. Using Pythagoras' Theorem the perpendicular height is 4. The area of the whole triangle is then $\frac{1}{2} \times 6 \times 4 = 12$.



6. **B** The sum of any three integers in arithmetic progression is a multiple of 3. For proof of this, if we let the smallest integer be a and the common difference of the sequence be d, then $a + (a+d) + (a+2d) = 3a + 3d = 3(a+d)$. As a result of the way the grid is filled, all the horizontal, vertical and diagonal lines contain numbers which are in arithmetic progression. Horizontally there are 2 lines of three cells in each of the 4 rows. Here $d = 1$. Vertically, there are again 2 lines in each of the 4 columns. Here $d = 4$. On the diagonals with positive gradient, there are 4 lines, with $d = -3$. On the diagonals with negative gradient there are four lines with $d = 5$. This is a total of $8 + 8 + 4 + 4 = 24$ lines.

7. **D** The sequence begins 2023, 2022, 1, 2021, 2020, 1, 2019, 2018, 1 … . Let the k^{th} term be u_k. Now consider the sequence $u_1, u_4, u_7, \ldots$, which starts 2023, 2021, 2019, … . Here the terms decrease by two each time. Since $25 = 1 + 8 \times 3$, $u_{25} = u_1 - 8 \times 2 = 2023 - 16 = 2007$.

8. **A** The value of $99(0.\dot{4}\dot{9} - 0.\dot{4}) = 99\left(\dfrac{49}{99} - \dfrac{4}{9}\right) = 99\left(\dfrac{49}{99} - \dfrac{44}{99}\right) = 99\left(\dfrac{49-44}{99}\right) = 99 \times \dfrac{5}{99} = 5$.

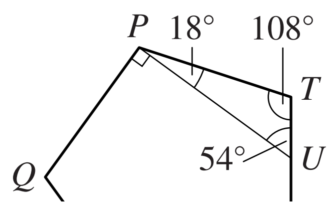
9. **D** For 1 Down, $2 \times 2^5 = 64$ is too small and $2 \times 4^5 = 2048$ is too big and therefore we must have $2 \times 3^5 = 486$. 3 Across must then start with a 6 and is therefore $5^4 = 625$. 2 Down must then end in a 5 and is therefore $5^3 = 125$. 1 Across is then 4 * 1. The only square of this form is $21^2 = 441$, so * is a 4.

10. **B** The sum $2^{2024} + 2^{2023} + 2^{2022}$ can be factorised to $2^{2022}(2^2 + 2^1 + 1) = 2^{2022} \times 7$. Hence, of the numbers listed, only 7 and $8 = 2^3$ are factors of 2^{2022}.

11. **B** Each of the four people is either telling the truth or lying. Assume first that Wenlu is telling the truth, then Xander is lying, which implies that Yasser is telling the truth which finally implies that Zoe is also telling the truth. In this case 3 people tell the truth. Now assume that Wenlu is lying. Therefore Xander is telling the truth that Yasser is lying and finally Zoe is also lying. In this case only 1 person tells the truth. In both cases, all four statements are consistent with each other.

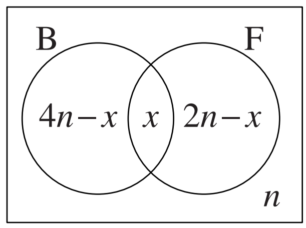
12. **E** As $n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n$, factors of 50! which contain a factor of 7 are 7, 14, 21, 28, 35, 42 and 49. The first six of these each contribute a single factor of 7 and 49 contributes two. The greatest power of 7 which is a factor of 50! is then 7^8, so $k = 8$.

13. **A** The interior angles of a regular pentagon are $180° - \frac{360°}{5} = 108°$. As $\angle QPU$ is a right angle, $\angle UPT = 108° - 90° = 18°$. As angles in a triangle sum to 180°, $\angle PUT = 180° - (108° + 18°) = 54°$. Therefore $\angle TPU : \angle PUT : \angle UTP = 18 : 54 : 108 = 1 : 3 : 6$.

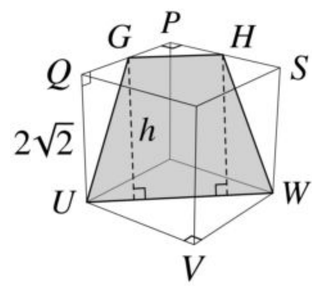


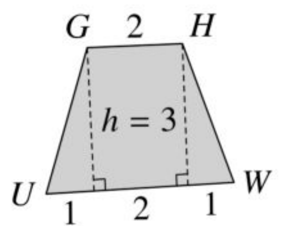
14. **E** The equation of the circle is $x^2 + y^2 = r^2$. At Q, $(12 - d)^2 + (2d - 6)^2 = r^2$. At P, $d^2 + (-d)^2 = r^2$, so $2d^2 = r^2$. Expanding the first equation and subtracting the second gives $144 - 24d + d^2 + 4d^2 - 24d + 36 - 2d^2 = 0$, which simplifies to $3d^2 - 48d + 180 = 0$. Dividing by 3 and factorising gives $(d-6)(d-10) = 0$. Therefore $d = 6$ or $d = 10$ and the sum of these values is 16.

15. **D** Let the number of people who play both basketball and football be x and the number who play neither be n. A Venn diagram can then be filled as shown. As there are 30 students, $7n - x = 30$. As $x \geq 0$, $7n - 30 \geq 0$ and so $n \geq 5$. From the Venn diagram it can be seen that $2n - x \geq 0$, therefore $2n - (7n - 30) \geq 0$ so $n \leq 6$. So $n = 5$ or 6 and the corresponding values of x are 5 or 12. The only one of these in the listed options is $x = 5$.



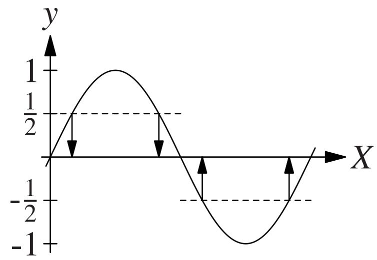
16. **A** To find the area of the trapezium, we require lengths of GH, UW and the perpendicular distance between them, h, say. In triangle PGH, $PG = PH = \sqrt{2}$ therefore $GH = \sqrt{\sqrt{2}^2 + \sqrt{2}^2} = 2$.
Triangle VUW is an enlargement of triangle PGH with scale factor 2, so $UW = 4$. In order to find h we must first find length GU. In triangle QUG, $UG = \sqrt{(2\sqrt{2})^2 + \sqrt{2}^2} = \sqrt{10}$.
From the triangular end of the trapezium, it follows that $1^2 + h^2 = \sqrt{10}^2$ therefore $h = 3$. The area of the trapezium $= \frac{1}{2} \times (4 + 2) \times 3 = 9$.





17. E In order to get the largest number, N, we need to make its earlier digits as large as possible, starting 9876 . . . as far as this works. However, since 53, 43, 23 and 13 are all prime, the digit 3 must precede all of 5, 4, 2 and 1. So the latest 3 can come is immediately after 6. Thereafter there are no reasons not to follow numerical order, making $N = 987635421$. Its 5^{th} and 6^{th} digits are 3 and 5.

18. C The equation $1 + 2\sin X - 4\sin^2 X - 8\sin^3 X = 0$ factorises to give $(1 + 2\sin X) - 4\sin^2 X(1 + 2\sin X) = 0$ and then to $(1 + 2\sin X)(1 - 4\sin^2 X) = 0$. Fully factorised, we have $(1 + 2\sin X)(1 + 2\sin X)(1 - 2\sin X) = 0$. So $\sin X = -\frac{1}{2}$ or $\sin X = \frac{1}{2}$. For $0° < X < 360°$, there are then four solutions as shown in the diagram.



19. E The expression $\dfrac{7n+12}{2n+3} \equiv \dfrac{4(2n+3)}{2n+3} - \dfrac{n}{2n+3} \equiv 4 - \dfrac{n}{2n+3}$. The first expression takes integer values precisely when $\dfrac{n}{2n+3}$ is an integer.

Consider first $n > 0$. When $n > 0$, $2n + 3 > n$, therefore $\dfrac{n}{2n+3} < 1$ so no integer values of the expression are possible.

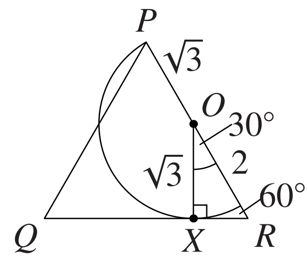
Next, consider $n = 0$. In this case, $\dfrac{n}{2n+3} = \dfrac{0}{0+3} = 0$ which is an integer.

When $n < 0$, in order to form an integer, we require $n \leq 2n + 3$, therefore $n \geq -3$.

Possible values of n are then $n = -1$, -2 and -3. The values of $\dfrac{n}{2n+3}$ in these cases are $\dfrac{-1}{2\times(-1)+3} = -1$, $\dfrac{-2}{2\times(-2)+3} = 2$ and $\dfrac{-3}{2\times(-3)+3} = 1$. Therefore the sum of the integer values of the initial expression is $(4 - 0) + (4 - (-1)) + (4 - 2) + (4 - 1) = 14$.

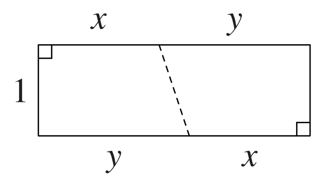
20. C Using Pythagoras' Theorem, $NM = \sqrt{(p+q)^2 + r^2}$ so the two journeys have lengths $p + r$ and $q + \sqrt{(p+q)^2 + r^2}$. Equating and rearranging, $p + r - q = \sqrt{(p+q)^2 + r^2}$ and so $(p + r - q)^2 = (p+q)^2 + r^2$. Expanding leads to $p^2 + 2pr - 2pq + r^2 - 2qr + q^2 = p^2 + 2pq + q^2 + r^2$ and therefore $2pr - 2qr = 4pq$. Rearranging to give q in terms of p and r, $pr = q(2p + r)$ so $q = \dfrac{pr}{2p+r}$.

21. D As the semicircle touches QR at X, the radius OX and tangent QR are perpendicular as shown. Triangle OXR is a 30°, 90°, 60° triangle and OX is given as $\sqrt{3}$. Therefore $XR = 1$ and $OR = 2$. As OP is also a radius of the circle, $OP = \sqrt{3}$ and $PR = QR = 2 + \sqrt{3}$. The length $QX = (2 + \sqrt{3}) - 1 = 1 + \sqrt{3}$.

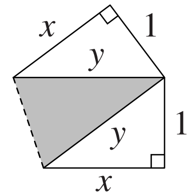


22. C Let $z = (\cos^{-1} x)$. Then $x = \cos z$ and $y = \sin z$ and therefore $x^2 + y^2 = 1$. As z lies between 0° and 180°, x lies between -1 and 1 and y lies between 0 and 1. Hence we get the upper semicircle shown on the graph in option C.

23. D In the first diagram shown, the paper is to be folded so that the bottom left vertex will lie on top of the top right vertex in order to form the desired pentagon. The fold line, shown dotted, must therefore lie on the perpendicular bisector of the line joining the bottom left and top right vertices and so pass through the centre of the rectangle. Labelling the longest sides of the rectangle with x and y,



$$x + y = 3. \qquad (1)$$



From the folded diagram, we have two right-angled triangles and in each, $1 + x^2 = y^2$. Rearranging and factorising gives

$$1 = (y + x)(y - x). \qquad (2)$$

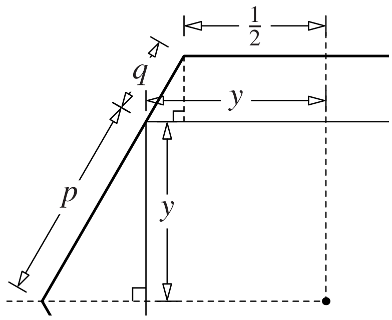
Substituting (1) into (2) gives $1 = 3(y - x)$ and so

$$\tfrac{1}{3} = y - x. \qquad (3)$$

Solving (1) and (3) leads to $y = \frac{5}{3}$ and $x = \frac{4}{3}$. The area of the pentagon = 1×3 – the shaded area. As the shaded area can be viewed as a triangle with base y and therefore perpendicular height 1, the area of the pentagon $= 3 - \frac{1}{2} \times y \times 1 = 3 - \frac{1}{2} \times \frac{5}{3} \times 1 = \frac{13}{6}$.

The area of the pentagon as a fraction of the area of the original rectangle is $\dfrac{\frac{13}{6}}{3} = \dfrac{13}{18}$.

24. B Let the square have side-length $2y$.
The two triangles shown, with hypotenuses p and q, have angles 30°, 60° and 90°. As the hexagon has side-length 1,



$$p + q = 1. \qquad (1)$$

From the larger triangle and from the top left of the square,

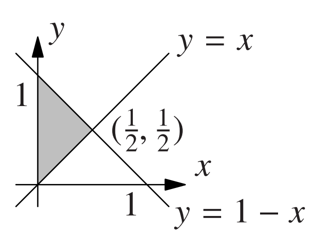
$$y = \frac{\sqrt{3}p}{2} \qquad \text{and} \qquad y = \tfrac{1}{2}q + \tfrac{1}{2}. \qquad (2)$$

Equating the two equations in (2) and rearranging gives $\sqrt{3}p - q = 1$. (3)
Solving (1) and (3) simultaneously gives $(\sqrt{3} + 1)p = 2$.
Rearranging and rationalising leads to

$$p = \frac{2}{(\sqrt{3}+1)}\frac{(\sqrt{3}-1)}{(\sqrt{3}-1)} = \sqrt{3} - 1.$$

Therefore, the length of the side of the square $2y = \sqrt{3}(\sqrt{3} - 1) = 3 - \sqrt{3}$.

25. A Factorising $x^3y^2 - x^2y^2 - xy^4 + xy^3 \geq 0$ gives $xy^2(x^2 - x - y^2 + y) \geq 0$. Rearranging to $xy^2(y - x - (y^2 - x^2)) \geq 0$ and then factorising gives $xy^2(y - x)(1 - y - x) \geq 0$. As $0 \leq x \leq y$, we know that $x \geq 0$, $y^2 \geq 0$ and $(y - x) \geq 0$ so the fourth factor, $(1 - y - x) \geq 0$. This rearranges to $y \leq 1 - x$. The lines $y = x$ and $y = 1 - x$ meet at $(\frac{1}{2}, \frac{1}{2})$ so the shaded region has area $\frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{4}$.



United Kingdom
Mathematics Trust

Senior Mathematical Challenge

Tuesday 4 October 2022

Organised by the United Kingdom Mathematics Trust

supported by Overleaf

Candidates must be full-time students at secondary school or FE college.
England & Wales: Year 13 or below
Scotland: S6 or below
Northern Ireland: Year 14 or below

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **90 minutes**.
   No answers, or personal details, may be entered after the allowed time is over.
3. The use of blank paper for rough working is allowed; **squared paper**, **calculators and measuring instruments are forbidden**.
4. **Use a B or an HB non-propelling pencil.** Mark A, B, C, D, E on the Answer Sheet for each question. Mark only one option, boldly, within the box.
5. Your Answer Sheet will be read by a machine. **Do not write or doodle on the sheet except to mark your chosen options.** The machine will read all markings, including bits of eraser stuck to the page, and interpret the mark in its own way.
6. **Do not expect to finish the whole paper in the time allowed.** The questions in this paper have been arranged in approximate order of difficulty with the harder questions towards the end. You are not expected to complete all the questions during the time. You should bear this in mind when deciding which questions to tackle.
7. **Scoring rules:** All candidates start with 25 marks; 0 marks are awarded for each question left unanswered; 4 marks are awarded for each correct answer; 1 mark is deducted for each incorrect answer (to discourage guessing).
8. **The questions on this paper are designed to challenge you to think, not to guess.** You will gain more marks, and more satisfaction, by doing one question carefully than by guessing lots of answers. This paper is about solving interesting problems, not about lucky guessing.
9. To accommodate candidates sitting at other times, please do not discuss the paper on the internet until 08:00 BST on Wednesday 5 October.

Enquiries about the Senior Mathematical Challenge should be sent to:

challenges@ukmt.org.uk

www.ukmt.org.uk

1. When the expression $\dfrac{(2^2-1)\times(3^2-1)\times(4^2-1)\times(5^2-1)}{(2\times3)\times(3\times4)\times(4\times5)\times(5\times6)}$ is simplified, which of the following is obtained?

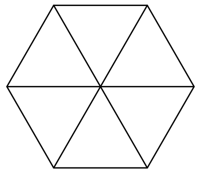
A $\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{4}$ D $\frac{1}{5}$ E $\frac{1}{6}$

2. What is the smallest prime which is the sum of five different primes?

A 39 B 41 C 43 D 47 E 53

3. The figure shows a regular hexagon.

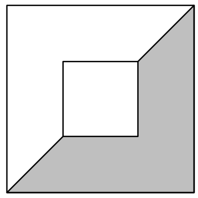
How many parallelograms are there in the figure?



A 2 B 4 C 6 D 8 E more than 8

4. The diagram shows two symmetrically placed squares with sides of length 2 and 5.

What is the ratio of the area of the small square to that of the shaded region?



A 7 : 24 B 1 : 3 C 8 : 25 D 8 : 21 E 2 : 5

5. What is the value of $\dfrac{1}{1.01}+\dfrac{1}{1.1}+\dfrac{1}{1}+\dfrac{1}{11}+\dfrac{1}{101}$?

A 2.9 B 2.99 C 3 D 3.01 E 3.1

6. What is the value of $\dfrac{4^{800}}{8^{400}}$?

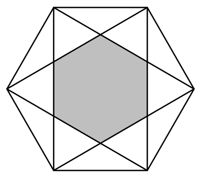
A $\frac{1}{2^{400}}$ B $\frac{1}{2^{200}}$ C 1 D 2^{200} E 2^{400}

7. In 2021, a first class postage stamp cost 85p and a second class postage stamp cost 66p. In order to spend an exact number of pounds and to buy at least one of each type, what is the smallest total number of stamps that should be purchased?

A 10 B 8 C 7 D 5 E 2

8. In the diagram, the outer hexagon is regular and has an area of 216.

What is the shaded area?



A 108 B 96 C 90 D 84 E 72

9. A light-nanosecond is the distance that a photon can travel at the speed of light in one billionth of a second. The speed of light is $3\times10^8\ \mathrm{ms}^{-1}$.

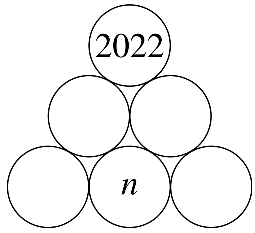
How far is a light-nanosecond?

A 3 cm B 30 cm C 3 m D 30 m E 300 m

10. What is the value of x in the equation $\dfrac{1+2x+3x^2}{3+2x+x^2}=3$?

A -5 B -4 C -3 D -2 E -1

11. In the number triangle shown, each disc is to be filled with a positive integer. Each disc in the top or middle row contains the number which is the product of the two numbers immediately below.



What is the value of n?

A 1 B 2 C 3 D 6 E 33

12. What is the sum of the digits of the integer which is equal to $6666666^2 - 3333333^2$?

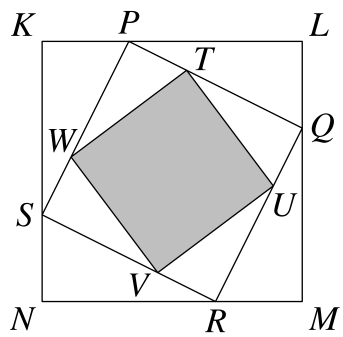
A 27 B 36 C 45 D 54 E 63

13. Three rugs have a combined area of $90\,\text{m}^2$. When they are laid down to cover completely a floor of area $60\,\text{m}^2$, the area which is covered by exactly two layers of rug is $12\,\text{m}^2$.

What is the area of floor covered by exactly three layers of rug?

A $2\,\text{m}^2$ B $6\,\text{m}^2$ C $9\,\text{m}^2$ D $10\,\text{m}^2$ E $12\,\text{m}^2$

14. The diagram shows a square, $KLMN$. A second square $PQRS$ is drawn inside it, as shown in the diagram, where P divides the side KL in the ratio $1 : 2$. Similarly, a third square $TUVW$ is drawn inside $PQRS$ with T dividing PQ in the ratio $1 : 2$.



What fraction of the area of $KLMN$ is shaded?

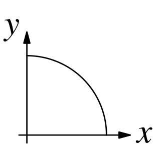
A $\frac{25}{81}$ B $\frac{16}{49}$ C $\frac{4}{9}$ D $\frac{40}{81}$ E $\frac{2}{3}$

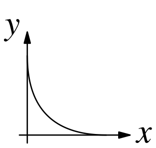
15. The hare and the tortoise had a race over 100 m, in which both maintained constant speeds. When the hare reached the finish line, it was 75 m in front of the tortoise. The hare immediately turned around and ran back towards the start line.

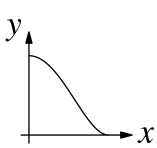
How far from the finish line did the hare and the tortoise meet?

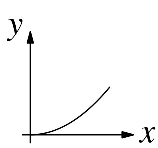
A 54 B 60 C 64 D $66\frac{2}{3}$ E 72

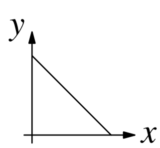
16. Which diagram could be a sketch of the curve $\sqrt{x} + \sqrt{y} = 1$?

A 

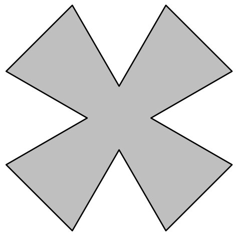
B 

C 

D 

E 

17. The shape shown is made by removing four equilateral triangles with side-length 1 from a regular octagon with side-length 1.



What is the area of the shape?

A $2 - 2\sqrt{2} + \sqrt{3}$ B $2 + 2\sqrt{2} - \sqrt{3}$ C $2 + 2\sqrt{2} + \sqrt{3}$
D $3 - 2\sqrt{2} - \sqrt{3}$ E $2 - 2\sqrt{2} - \sqrt{3}$

18. The numbers x and y are such that $3^x + 3^{y+1} = 5\sqrt{3}$ and $3^{x+1} + 3^y = 3\sqrt{3}$.

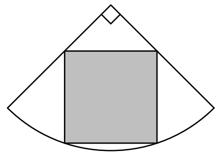
What is the value of $3^x + 3^y$?

A $\sqrt{3}$ B $2\sqrt{3}$ C $3\sqrt{3}$ D $4\sqrt{3}$ E $5\sqrt{3}$

19. How many pairs of real numbers (x, y) satisfy the simultaneous equations $x^2 - y = 2022$ and $y^2 - x = 2022$?

A infinitely many B 1 C 2 D 3 E 4

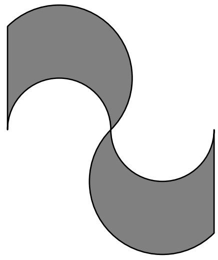
20. A square is inscribed inside a quadrant of a circle. The circle has radius 10.



What is the area of the square?

A $25\sqrt{2}$ B 36 C 12π D 40 E $30\sqrt{2}$

21. The perimeter of a logo is created from two vertical straight edges, two small semicircles with horizontal diameters and two large semicircles. Both of the straight edges and the diameters of the small semicircles have length 2. The logo has rotational symmetry as shown.



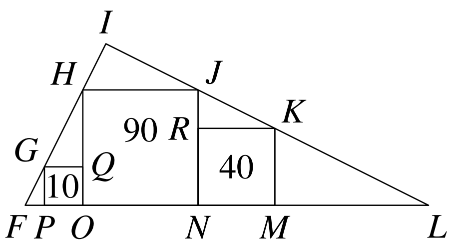
What is the shaded area?

A 4 B $4 - \pi$ C 8 D $4 + \pi$ E 12

22. How many pairs of integers (x, y) satisfy the equation $\sqrt{x - \sqrt{x + 23}} = 2\sqrt{2} - y$?

A 0 B 1 C 4 D 8 E infinitely many

23. Three squares $GQOP$, $HJNO$ and $RKMN$ have vertices which sit on the sides of triangle FIL as shown. The squares have areas of 10, 90 and 40 respectively.



What is the area of triangle FIL?

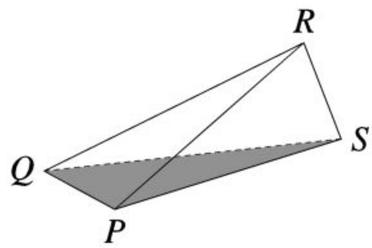
A 220.5 B $\frac{21}{5}\sqrt{10}$ C 252 D $\frac{21}{2}\sqrt{10}$ E 441

24. The numbers x, y, p and q are all integers. x and y are variable and p and q are constant and positive. The four integers are related by the equation $xy = px + qy$.

When y takes its maximum possible value, which expression is equal to $y - x$?

A $pq - 1$ B $(p - 1)(q - 1)$ C $(p + 1)(q - 1)$ D $(p - 1)(q + 1)$ E $(p + 1)(q + 1)$

25. A drinks carton is formed by arranging four congruent triangles as shown. $QP = RS = 4$ cm and $PR = PS = QR = QS = 10$ cm.



What is the volume, in cm^3, of the carton?

A $\frac{16}{3}\sqrt{23}$ B $\frac{4}{3}\sqrt{2}$ C $\frac{128}{25}\sqrt{6}$ D $\frac{13}{2}\sqrt{23}$ E $\frac{8}{3}\sqrt{6}$

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For reasons of space, these solutions are necessarily brief.

There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation.

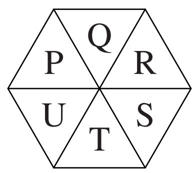
There is also a version of this document available on the UKMT website which includes each of the questions alongside its solution:

www.ukmt.org.uk

1. D The expression simplifies to $\dfrac{3 \times 8 \times 15 \times 24}{(2 \times 3) \times (3 \times 4) \times (4 \times 5) \times (5 \times 6)}$. Cancelling common factors gives $\frac{1}{5}$.

2. C For the sum of five different primes to be prime, each of those five primes must be odd, Listing the primes starting with 3, 5, 7, 11, 13, 17, 19, . . . and working systematically through possible sums gives a smallest sum of $3 + 5 + 7 + 11 + 13 = 39$ which is not prime. However, the next smallest sum $3 + 5 + 7 + 11 + 17 = 43$ which is prime as required.

3. C Each of the possible parallelograms is formed from two adjacent equilateral triangles: P and Q, Q and R, R and S, S and T, T and U and finally U and P. Therefore there are six possible parallelograms.



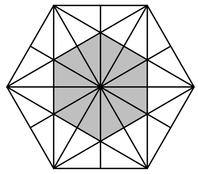
4. D The area of the small square is $2 \times 2 = 4$. The area of the shaded region is then $\frac{1}{2} \times 5 \times 5 - \frac{1}{2} \times 2 \times 2 = \frac{25-4}{2} = \frac{21}{2}$. Therefore the ratio of the area of the small square to the area of the shaded region is $4 : \frac{21}{2} = 8 : 21$.

5. C Rewriting the calculation as $\frac{100}{101} + \frac{10}{11} + \frac{1}{1} + \frac{1}{11} + \frac{1}{101}$ shows that we can reorder the sum to give $\frac{101}{101} + \frac{11}{11} + \frac{1}{1} = 3$.

6. E Rewriting $\dfrac{4^{800}}{8^{400}}$ using a base of 2 gives $\dfrac{(2^2)^{800}}{(2^3)^{400}} = \dfrac{2^{1600}}{2^{1200}} = 2^{400}$ using rules of indices.

7. C Consider first the units digits of 85 and 66. Multiples of 5 can only end in 5 or 0. No multiples of 6, an even number, can end in 5. So in order that the units digit of our sum can be 0, each of the multiples of 85 and 66 must individually have units digits of 0. The smallest multiples of 85 and 66 with this property, $2 \times 85 = 170$ and $5 \times 66 = 330$, have sum 500. So 7 is the smallest number of stamps and they cost £5.

8. E By drawing extra lines from the centre of the outer hexagon to each of its vertices and from the centre to the midpoint of each edge of the outer hexagon, 12 in total, the diagram can be shown to be made of 36 congruent triangles each with angles 30°, 60° and 90°. Twelve of these triangles are shaded giving a shaded area of $\frac{1}{3} \times 216 = 72$.



9. B Using speed $= \dfrac{\text{distance}}{\text{time}}$ gives $3 \times 10^8 = \dfrac{d}{10^{-9}}$. Therefore $d = 3 \times 10^{-1}$ m $= 0.3$ m $= 30$ cm.

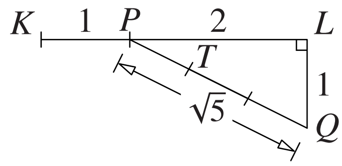
10. D Rearranging the equation gives $1 + 2x + 3x^2 = 9 + 6x + 3x^2$ so $1 + 2x = 9 + 6x$ and $4x = -8$. Therefore $x = -2$.

11. A When expressed as the product of its prime factors, $2022 = 2 \times 3 \times 337$. However, the integer n must be a factor of each integer in the middle row and so n^2 must be a factor of their product 2022. Therefore $n = 1$.

12. E Using the difference of two squares, the calculation we are given can be written in the form $6666666^2 - 3333333^2 = (6666666 + 3333333)(6666666 - 3333333) = 9999999 \times 3333333 = 10000000 \times 3333333 - 1 \times 3333333 = 33333330000000 - 3333333 = 33333326666667$. The sum of the digits of this integer is 63.

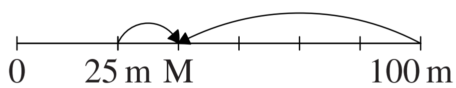
13. C Let the area of floor covered by exactly one rug be a, the area of floor covered by exactly two rugs be b and the area of floor covered by three rugs be c. Therefore, $a + 2b + 3c = 90$ and $a + b + c = 60$. Subtracting the second equation from the first leaves $b + 2c = 30$ and using $b = 12$ gives $c = 9$.

14. A Let KL be 3 units long. Then $KP = 1$, $PL = 2$ and area $KLMN = 3 \times 3 = 9$. Removing four right-angled triangles congruent to PLQ from square $KLMN$ gives area $PQRS = 9 - 4 \times \frac{1}{2} \times 1 \times 2 = 5$.The area of $PQRS$ is $\frac{5}{9}$ of the area of $KLMN$. By the same reasoning the area of $TUVW$ is $\frac{5}{9}$ of the area of $PQRS$.



Combining these proportions gives the shaded area as $\frac{5}{9} \times \frac{5}{9} = \frac{25}{81}$ of the area of $KLMN$.

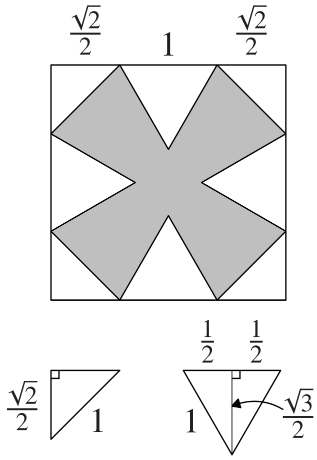
15. B When the hare and tortoise are moving in the same direction, the hare completes 100 m while the tortoise completes 25 m. After the hare reverses direction and the hare and tortoise are moving towards one another, the hare is still moving four times as fast.



Therefore the meeting point, M, is $\frac{4}{5}$ of 75 m = 60 m away from the finish line.

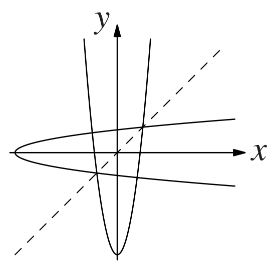
16. B As x and y are interchangeable in the equation, the graph must be symmetric about the line $y = x$. This excludes options C and D. Substituting $x = 0$ and $x = 1$ into the equation shows that the graph crosses the axes at $(0, 1)$ and $(1, 0)$. Note that in option E the line $x + y = 1$ meets $y = x$ at $(\frac{1}{2}, \frac{1}{2})$ whereas our curve meets $y = x$ at $(\frac{1}{4}, \frac{1}{4})$ and must therefore lie below the straight line shown in option E. The only possible option then is B.

17. B We enclose the regular octagon within a square as shown. Since the side-length of the octagon is 1, the right-angled isosceles triangles in the corners have two short sides of length $\frac{\sqrt{2}}{2}$ and so the square has side-length $1 + \sqrt{2}$. Each of the right-angled triangles has area $\frac{1}{2} \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} = \frac{1}{4}$. Each of the equilateral triangles which were removed has base 1 and so height $\frac{\sqrt{3}}{2}$. The shaded area can be obtained as the area of the square minus that of the four isosceles corners and the four equilateral triangles; that is $(1+\sqrt{2})^2 - 4 \times \frac{1}{4} - 4 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = 3 + 2\sqrt{2} - 1 - \sqrt{3} = 2 + 2\sqrt{2} - \sqrt{3}$.

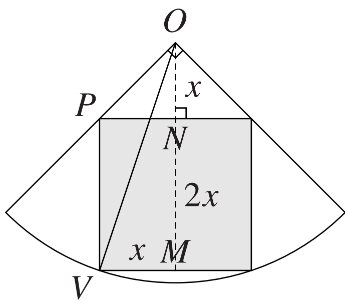


18. B Let $3^x = X$ and $3^y = Y$. The two equations can then be written as $X + 3Y = 5\sqrt{3}$ and $3X + Y = 3\sqrt{3}$. Subtracting three lots of the second equation from the first gives $-8X = -4\sqrt{3}$ so $X = \frac{\sqrt{3}}{2}$. Subtracting three lots of the first equation from the second gives $-8Y = -12\sqrt{3}$ so $Y = \frac{3\sqrt{3}}{2}$. The value of $3^x + 3^y = X + Y = \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{2} = 2\sqrt{3}$. Alternatively, we could add the two equations giving $4X + 4Y = 8\sqrt{3}$. Dividing by 4, $X + Y = 3^x + 3^y = 2\sqrt{3}$ without knowing the value of either 3^x or 3^y individually.

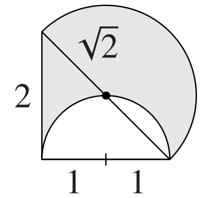
19. E The first equation can be rearranged to the form $y = x^2 - 2022$ which is a translation of $y = x^2$ down 2022 units. The second equation is a reflection of the first, in the line $y = x$. There are four points of intersection of these two parabolas.



20. D Let O be the centre of the circle, M and N be the midpoints of two sides of the square and V and P be vertices of two sides of the square as shown. Line ONM is a line of symmetry. Let $ON = x$. Therefore $NP = x$ as ON and NP are sides of the right-angled isosceles triangle ONP. Also, $PV = MN = 2x$. Consider right-angled triangle OVM. The radius of the circle is given as 10, therefore $(3x)^2 + x^2 = 10^2$ so $10x^2 = 100$ and $x^2 = 10$. Hence the area of the square is $(2x)^2 = 4x^2 = 4 \times 10 = 40$.

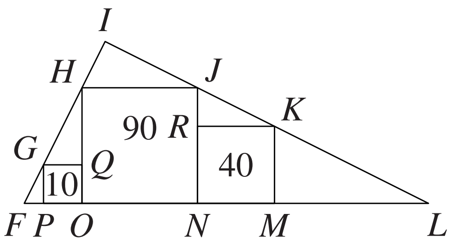


21. D Half the diagram is shown here. In it, the shaded area equals the area of a right-angled isosceles triangle of side-length 2 plus the area of a large semicircle minus the area of a small semicircle of radius 1. Using Pythagoras' Theorem, the diameter of the large semicircle has length $2\sqrt{2}$ and so the radius is $\sqrt{2}$. Therefore the shaded area of the full diagram is $2[\frac{1}{2} \times 2 \times 2 + \frac{1}{2}\pi \times (\sqrt{2})^2 - \frac{1}{2}\pi \times 1^2] = 2(2 + \pi - \frac{1}{2}\pi) = 4 + \pi$.



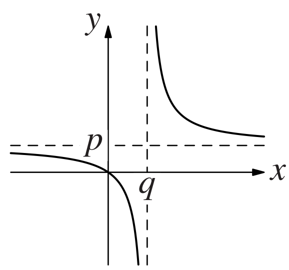
22. B Squaring both sides of the equation gives $x - \sqrt{x+23} = 8 - 4\sqrt{2}y + y^2$ which can be rearranged to $\sqrt{x+23} = (x - 8 - y^2) + 4\sqrt{2}y$ [1]. Squaring equation [1] gives $x + 23 = (x - 8 - y^2)^2 + 2(4\sqrt{2}y)(x - 8 - y^2) + 32y^2$ [2]. We are given that both x and y are integers and so the surd component, $2(4\sqrt{2}y)(x - 8 - y^2)$, must equal 0. Therefore either $y = 0$ or $(x - 8 - y^2) = 0$ [3]. Consider first the case $y = 0$. Here, equation [2] reduces to $x + 23 = (x - 8)^2$. This expands to $x^2 - 17x + 41 = 0$ which has no integer solutions as its discriminant is $(-17)^2 - 4 \times 1 \times 41 = 125$, which is not square. Secondly considering $(x - 8 - y^2) = 0$ [3] reduces [1] to $\sqrt{x+23} = 4\sqrt{2}y$ and therefore $x + 23 = 32y^2$. Using [3] again gives $x = 8 + y^2$ and so $31 + y^2 = 32y^2$. Therefore $y^2 = 1$. Hence $y = \pm 1$ and in either case, $x = 8 + 1 = 9$. Because equations have been squared, some solutions could be spurious. Substituting in the original equation, we see that $(9, 1)$ is a solution but $(9, -1)$ is not. Hence there is just one solution.

23. A The lengths of the sides of the three squares are $\sqrt{10}$, $3\sqrt{10}$ and $2\sqrt{10}$ respectively. Therefore $HQ = 2\sqrt{10}$ and $RJ = \sqrt{10}$. In triangle GQH, the gradient of GH is $\frac{2\sqrt{10}}{\sqrt{10}} = 2$. In triangle JRK, the gradient of JK is $\frac{-\sqrt{10}}{2\sqrt{10}} = \frac{-1}{2}$. Therefore lines FI (on which GH lies) and IL (on which JK lies) are perpendicular.

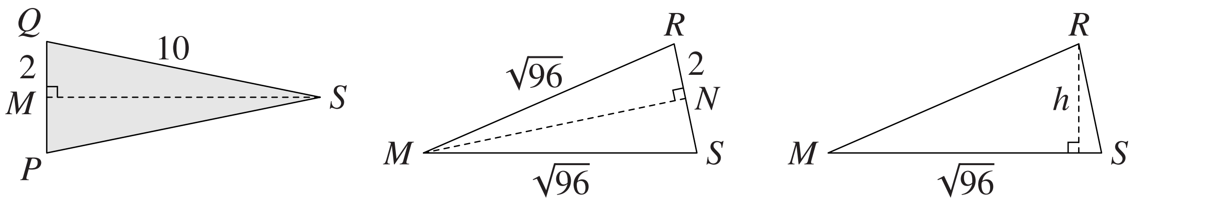


All five right-angled triangles around the edge of the figure and triangle FIL itself are similar as they contain the same angles. They all have sides in the ratio $1 : 2 : \sqrt{5}$. To calculate the area of triangle FIL we need the length IL, as the area of $FIL = \frac{1}{2} \times IL \times \frac{1}{2}IL$. The length IL is made of three sections: $JK = \sqrt{10} \times \sqrt{5}$, $KL = 2JK = 2 \times \sqrt{10} \times \sqrt{5}$ and $IJ = \frac{2}{\sqrt{5}} \times HJ = \frac{2}{\sqrt{5}} \times 3\sqrt{10}$. Therefore $IL = IJ + JK + KL = 6\sqrt{2} + \sqrt{50} + 2\sqrt{50} = 21\sqrt{2}$. Hence the area of triangle $FIL = \frac{1}{2} \times 21\sqrt{2} \times \frac{21}{2}\sqrt{2} = 220.5$.

24. D Rearranging $xy = px + qy$ to make y the subject, gives $xy - qy = px$ so $y(x - q) = px$ and therefore $y = \frac{px}{x-q}$ which rearranges to $y = p + \frac{pq}{x-q}$. A sketch of the graph of this function for real values of x and y is shown. As x and y are both integers in this question, y takes its maximum value when $x - q$ is as small as possible therefore $x - q = 1$ so $x = q + 1$. The expression $y - x$ then becomes $\dfrac{px}{1} - x = (p - 1)x = (p - 1)(q + 1)$.



25. A



Let M be the midpoint of QP. The volume of the carton is $\frac{1}{3} \times$ base area of triangle $PQS \times$ the perpendicular height from R to the plane containing PQS. Triangle PQS is isosceles and $MS = \sqrt{10^2 - 2^2} = \sqrt{96}$. So area of $PQS = \frac{1}{2} \times 4 \times \sqrt{96} = 8\sqrt{6}$. Consider isosceles triangle MRS and let N be the midpoint of RS. $MN = \sqrt{(\sqrt{96})^2 - 2^2} = \sqrt{92}$, so with RS as the base, area of $MRS = \frac{1}{2} \times 4 \times \sqrt{92} = 4\sqrt{23}$. Now with MS as the base, area of $MRS = \frac{1}{2} \times MS \times h$. Therefore $4\sqrt{23} = \frac{1}{2} \times \sqrt{96} \times h$ and $h = \frac{2\sqrt{23}}{\sqrt{6}}$. Finally, the volume $= \frac{1}{3} \times 8\sqrt{6} \times \frac{2\sqrt{23}}{\sqrt{6}} = \frac{16\sqrt{23}}{3}$.